## James boundaries and $\sigma$ -fragmented selectors

## B. Cascales

Universidad de Murcia

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## The co-authors

B. C and **M. Muñoz and J. Orihuela**, James boundaries and  $\sigma$ -fragmented selectors, Preprint. 2007. Available at http://misuma.um.es/beca



B. C, V. Fonf, J. Orihuela, and S. Troyanski, *Boundaries in Asplund spaces*, Preprint 2007.

- 1 Two problems about boundaries
- 2 Some old results about boundaries and compactness

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- 3 Some new results about boundaries and selectors
- Open problems

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Boundaries <sup>,</sup> defi	nitions		

- X is a Banach space equipped with its norm || ||;
- K is a Hausdorff compact and C(K) is equipped with its supremum norm.

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- A subset  $B \subset B_{X^*} = \{x^* \in X^* ; \|x^*\| \le 1\}$  is a boundary for  $B_{X^*}$  if for any  $x \in X$ , there is  $x^* \in B$  such that  $x^*(x) = \|x\|$ .

2 problems about boundaries	The Boundary problem	Our new results	Open problems+References
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2 problems ab	out boundaries	The Boundary problem	Our new results	Open problems+References
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## Two problems regarding boundaries

## Problem 1: The boundary problem (Godefroy)...extremal test

Let X Banach space,  $B \subset B_{X^*}$  boundary and denote by  $\tau_p(B)$  the topology defined on X by the pointwise convergence on B. Let H be a norm bounded and  $\tau_p(B)$ -compact subset of X.

Is H weakly compact?

2 problems 0●0	s about boundaries	The Boundary problem 0	Our new results 00000	Open problems+References
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## Problem 2: When is a boundary strong?

Let X Banach space,  $B \subset B_{X^*}$  boundary.

When do we have  $B_{X^*} = \overline{\operatorname{coB}}^{\| \|}$ ?

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2 problems about boundaries	The Boundary problem	Our new results	Open problems+References
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Two problems re	garding bounda	aries	

## Let X Banach space, $B \subset B_{X^*}$ boundary and $H \subset X$ norm bounded.

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Is *H* weakly compact?

$$B_{X^*} = \overline{\operatorname{co} B}^{\| \|}?$$

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1952, Grothendieck: X = C(K) and  $B = \text{Ext}(B_{C(K)^*});$ 

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1976, Haydon, YES:  $\ell^1 \not\subset X$  and  $B = \operatorname{Ext} B_{X^*}$ 

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- 2003, Fonf-Lindenstrauss: alternative proofs.

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Right  $\implies$  Left. Left is open in full generality. Right isn't always true.

2 problems about boundaries	The Boundary problem •	Our new results 00000	Open probl
Boundary problem	n for $C(K)$		

#### G. Godefroy and B. C., 1998

Let K be a compact space and  $B \subset B_{C(K)^*}$  a boundary. Then a subset H of C(K) is weakly compact if, and only if, it is norm bounded and  $\tau_p(B)$ -compact.

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## Boundary problem for C(K)

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Let X be a Banach space such that  $\ell^1(c) \notin X$  and B any boundary for  $B_{X^*}$ . Then a subset H of X is weakly compact if, and only if, it is norm bounded and  $\tau_p(B)$ -compact.

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2 problems about boundaries	0 O	Our new results ●○○○○	Open problems+Referenc	es
Strong boundar	ies			
Definition				
Given a Banach sp for $K$ is a subset $E$ such that $b(x) = s$	ace X and a w <sup>*</sup> -compace B of K such that for even $\{k(x) : k \in K\}.$	ct subset $K \subset X^*$ , a ry $x \in X$ there exist	James boundary is some $b \in B$	

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	Definition			
	Given a Banach space for $K$ is a subset $B$ of such that $b(x) = \sup \{$	X and a w <sup>*</sup> -compact K such that for every $\{k(x) : k \in K\}$ . If K is	subset $K \subset X^*$ , a Jack $x \in X$ there exists a convex then $K = \overline{co}$	ames boundary some $b \in B$ $\overline{B}^{W^*}$ .

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The question?			
$K \text{ is convex, } B$ $K = \overline{\operatorname{co} B}^{\parallel \parallel}.$	$\subset K$ boundary, study cond	litions (X, B or K?	) leading to

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2 prot 000	olems about boundaries	The Boundary problem O	Our new results	Open problems+References	
Str	rong boundaries	5			
	Definition				
	Given a Banach space X and a w <sup>*</sup> -compact subset $K \subset X^*$ , a James boundary for K is a subset B of K such that for every $x \in X$ there exists some $b \in B$ such that $b(x) = \sup \{k(x) : k \in K\}$ . If K is convex then $K = \overline{\operatorname{co} B}^{w^*}$ .				
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	$K$ is convex, $B \subset K$ b $K = \overline{\operatorname{co} B}^{\parallel} \parallel$ .	oundary, study condi	tions (X, B or K?)	leading to	
	What are the technique 1976 Haydon [b	ues that have been us $\frac{1}{2} d = \frac{1}{2} d = \frac{1}{2$	sed? 3 — Fxt K uses inde	nendent	

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■ 1976, Haydon [Hay76]:  $\ell^1 \not\subset X$  and B = ExtK uses independent sequences (Ramsey theory)  $K = \overline{\text{coExt}K}^{\parallel \parallel}$ .

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K is convex $B \subset K$	boundary study condi	$(X B \text{ or } K^2)$	leading to			

 $K = \overline{\operatorname{co} B}^{\parallel \parallel}$ 

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- **2** 1987, Namioka [Nam87]:  $K \subset X^*$  is norm fragmented, then  $\overline{\operatorname{co} K}^{w^*} = \overline{\operatorname{co} K}^{\| \|}$ , uses the existence of barycenters.
- **3** 1987, Godefroy [God87]: if  $B \subset K$  is norm separable then  $K = \overline{\operatorname{co} B}^{\| \cdot \|}$  uses *Simons inequality*.

2 prob 000	olems about boundaries	The Boundary problem O	Our new results ●○○○○	Open problems+References 00
Str	rong boundaries	;		
	Definition			
	Given a Banach space for $K$ is a subset $B$ of such that $b(x) = \sup \{$	X and a w*-compact s K such that for every $x \in K$ ( $k(x) : k \in K$ ). If K is c	ubset $K \subset X^*$ , a Jack $K \in X$ there exists so onvex then $K = \overline{co}$	$\begin{array}{l} \text{mes boundary} \\ \text{ome } b \in B \\ \overline{B}^{w^*}. \end{array}$
	The question?			

*K* is convex,  $B \subset K$  boundary, study conditions (*X*, *B* or *K*?) leading to  $K = \overline{\operatorname{co} B}^{\parallel \parallel}$ .

### What are the techniques that have been used?

- I1976, Haydon [Hay76]: l<sup>1</sup> ∉ X and B = Ext K uses independent sequences (Ramsey theory) K = coExt K<sup>||</sup> ||.
- **2** 1987, Namioka [Nam87]:  $K \subset X^*$  is norm fragmented, then  $\overline{\operatorname{co} K}^{w^*} = \overline{\operatorname{co} K}^{\| \|}$ , uses the existence of barycenters.
- **3** 1987, Godefroy [God87]: if  $B \subset K$  is norm separable then  $K = \overline{\operatorname{co} B}^{\| \|}$  uses *Simons inequality*.
- 1987, Godefroy [God87] using Simons inequality proves that if X is separable and  $\ell^1 \not\subset X$  then  $K = \overline{\operatorname{co}B}^{\parallel \parallel}$ .

2 problems about boundaries	The Boundary problem	Our new results	Open problems+References
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Our results			

• We prove that when *B* is "descriptive" then  $K = \overline{\operatorname{co} B}^{\| \|}$ : this extends results by Godefroy, Contreras-Payá and *solve* a problem asked by Plichko.

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We apply the techniques developed to give new characterizations of Asplund spaces.

2 problems about boundaries	The Boundary problem 0	Our new results	Open problems+References 00
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- **(**) We prove that when *B* is "descriptive" then  $K = \overline{\operatorname{co} B}^{\| \|}$ : this extends results by Godefroy, Contreras-Payá and *solve* a problem asked by Plichko.
- We apply the techniques developed to give new characterizations of Asplund spaces.
- We prove that Fonf-Lindenstrauss techniques can be reduced to the *old techniques* coming from Simons inequality: there are no new techniques nor can be stronger applications derived from Fonf-Lindenstrauss.

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- We characterize Banach spaces X without copies of l<sup>1</sup> via boundaries extending the results by Godefroy for the separable case.

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- **(**) We prove that when *B* is "descriptive" then  $K = \overline{\operatorname{co} B}^{\| \|}$ : this extends results by Godefroy, Contreras-Payá and *solve* a problem asked by Plichko.
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- We characterize Banach spaces X without copies of l<sup>1</sup> via boundaries extending the results by Godefroy for the separable case.
- So For Asplund spaces we characterize boundaries for which K = co B<sup>||||</sup>. We extend in several different ways results by Namioka and Fonf.



## Proposition, Muñoz-Orihuela-B.C.

Let X be a Banach space, B a boundary for  $B_{X^*}$ ,  $1 > \varepsilon \ge 0$  and  $T \subset X^*$  such that  $B \subset \bigcup_{t \in T} B(t, \varepsilon)$ . If (T, w) is countably K-determined (resp. K-analytic) then:

(i)  $X^* = \overline{\text{span } T}^{\parallel \parallel}$  and  $X^*$  is weakly countably *K*-determined (resp. weakly *K*-analytic).

(ii) Every boundary for  $B_{X^*}$  is strong. In particular  $B_{X^*} = \overline{\operatorname{co}(B)}^{\parallel \parallel}$ .



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This answers a question by Plichko, extends Godefroy's result for separable boundary and improves Contreras-Payá and Fonf-Lindenstrauss result.

2 problems about bound 000	aries The Boundary prol O	olem Our new r	esults Op oc	pen problems+References
2nd result:	characterizatior	n of Asplund	spaces v	via selectors

### Muñoz-Orihuela-B.C.

The following conditions are equivalent for a Banach space X:

- (i) X is an Asplund space;
- (ii) J has a Baire one selector;
- (iii) J has a  $\sigma$ -fragmented selector;
- (iv) for some  $0 < \varepsilon < 1$ , J has an  $\varepsilon$ -selector that sends norm separable subsets of X into norm separable subsets of X\*.
- (v) there exists  $0 < \varepsilon < 1$  such that  $(B_{X^*}, w^*)$  is  $\varepsilon$ -fragmented, *i.e.*, for every non-empty subset  $C \subset B_{X^*}$  there exists some w\*-open set V in  $B_{X^*}$  such that  $C \cap V \neq \emptyset$  and  $\| \| \operatorname{diam}(C \cap V) < \varepsilon$ .

2	problems	about	boundaries	

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### Duality mapping

If  $(X, \| \|)$  is a Banach space the duality mapping  $J: X \to 2^{B_{X^*}}$  is defined at each  $x \in X$  by

$$J(x) := \{x^* \in B_{X^*} : x^*(x) = ||x||\}.$$

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### Notes:

 $\bullet\,$  Borel measurable maps are  $\sigma\text{-fragmented}.$ 

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- Borel measurable maps are  $\sigma$ -fragmented.
- The implication (ii)⇒(i) is proved in [JR02] with extra hypothesis which are justified with a wrong example.

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### Notes:

- Borel measurable maps are σ-fragmented.
- The implication (ii)⇒(i) is proved in [JR02] with extra hypothesis which are justified with a wrong example.
- The equivalence with (v) is known when we write *for every* ε: a different proof has been given quite recently by Fabian-Montesinos-Zizler.

2 problems about boundaries	The Boundary problem	Our new results	Open problems+References
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Boundaries and th	he topology $\gamma$		

 $\gamma$  is the topology on  $X^*$  of uniform convergence on bounded and countable subsets of X.

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2 problems about boundaries	The Boundary problem 0	Our new results	Open problems+References 00
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## Boundaries and the topology $\gamma$

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### Muñoz, Orihuela and B. C.

Let X be a Banach space. The following statement are equivalent:

- (i)  $\ell^1 \not\subset X$ ;
- (ii) for every w\*-compact subset K of X\* and any boundary B of K we have  $\overline{\operatorname{co}(K)}^{w^*} = \overline{\operatorname{co}(B)}^{\gamma}$ ;

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2 problems about boundaries	The Boundary problem	Our new results	Open problems+References
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### Fonf, Troyanski, Orihuela and B. C.

Let X be an Asplund space, K a w<sup>\*</sup>-compact convex subset of the dual space  $X^*$  and  $B \subset K$  a boundary of K. Each one of the condition below implies that  $K = \overline{\operatorname{co} B}^{\parallel \parallel}$ :

- (i) B is  $\gamma$ -closed.
- (ii) B is w<sup>\*</sup>-K-analytic.

2 problems about boundaries Open problem Our new results Open problems+References oo oo

## Boundaries and the topology $\gamma$

The techniques now are *topological* techniques developed by Namioka-Orihuela-B. C and Namioka-B. C.

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Let X be a Banach space. The following statement are equivalent:

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2 problems about boundaries	The Boundary problem	Our new results	Open problems+References
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Two open proble	mc		

- The boundary problem in full generality (Godefroy).
- Output Characterize strong boundaries out of the setting of Asplund spaces.

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2 problems about boundaries	The Boundary problem	Our new results	Open problems+References
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